

# Reflection positivity for the circle group

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**Abstract.** In this note we characterize those unitary one-parameter groups  $(U_t^c)_{t \in \mathbb{R}}$  which admit euclidean realizations in the sense that they are obtained by the analytic continuation process corresponding to reflection positivity from a unitary representation  $U$  of the circle group. These are precisely the ones for which there exists an anti-unitary involution  $J$  commuting with  $U^c$ . This provides an interesting link with the modular data arising in Tomita–Takesaki theory. Introducing the concept of a positive definite function with values in the space of sesquilinear forms, we further establish a link between KMS states and reflection positivity on the circle.

## 1. Introduction

In this note we continue our investigations of the mathematical foundations of *reflection positivity*, a basic concept in constructive quantum field theory ([GJ81, JO198, JO100, JR08, JR07a, JR07b]). Originally, reflection positivity, also called Osterwalder–Schrader positivity, arises as a requirement on the euclidean side to establish a duality between euclidean and relativistic quantum field theories ([OS73]). It is closely related to “Wick rotations” or “analytic continuation” in the time variable from the real to the imaginary axis.

The underlying fundamental concept is that of a *reflection positive Hilbert space*, introduced in [NO14a]. This is a triple  $(\mathcal{E}, \mathcal{E}_+, \theta)$ , where  $\mathcal{E}$  is a Hilbert space,  $\theta : \mathcal{E} \rightarrow \mathcal{E}$  is a unitary involution and  $\mathcal{E}_+$  is a closed subspace of  $\mathcal{E}$  which is  $\theta$ -positive in the sense that the hermitian form  $\langle \theta u, v \rangle$  is positive semidefinite on  $\mathcal{E}_+$ .<sup>1</sup> Let  $\mathcal{N} := \{v \in \mathcal{E}_+ : \langle \theta v, v \rangle = 0\}$ , write  $\widehat{\mathcal{E}}$  for the Hilbert space completion of the quotient  $\mathcal{E}_+/\mathcal{N}$  and  $q : \mathcal{E}_+ \rightarrow \widehat{\mathcal{E}}, v \mapsto \widehat{v}$  for the canonical map. If  $T : \mathcal{D}(T) \subseteq \mathcal{E}_+ \rightarrow \mathcal{E}_+$  is a linear or antilinear operator with  $T(\mathcal{N} \cap \mathcal{D}(T)) \subseteq \mathcal{N}$ , then there exists a well-defined operator  $\widehat{T} : \mathcal{D}(\widehat{T}) \subseteq \widehat{\mathcal{E}} \rightarrow \widehat{\mathcal{E}}$  defined by  $\widehat{T}(\widehat{v}) = \widehat{Tv}$  for  $v \in \mathcal{D}(T)$ . The  $\widehat{\phantom{x}}$  is the *OS-quantization functor*.

To see how this relates to group representations, let us call a triple  $(G, H, \tau)$  a *symmetric Lie group* if  $G$  is a Lie group,  $\tau$  is an involutive automorphism of  $G$  and  $H$  is an open subgroup of the group  $G^\tau$  of  $\tau$ -fixed points. Then the Lie algebra  $\mathfrak{g}$  of  $G$  decomposes into  $\tau$ -eigenspaces  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$  and we obtain the *Cartan dual Lie algebra*  $\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}$ . If  $(G, H, \tau)$  is a symmetric Lie group and  $(\mathcal{E}, \mathcal{E}_+, \theta)$  a reflection positive Hilbert space, then we say that a unitary representation  $\pi : G \rightarrow \mathcal{E}$  is *reflection positive with respect to*  $(G, H, \tau)$  if the following three conditions hold:

(RP1)  $\pi(\tau(g)) = \theta\pi(g)\theta$  for every  $g \in G$ .

<sup>1</sup> We write  $\langle v, w \rangle$  for the scalar product in a complex Hilbert space and we assume that it is linear in the *first* argument.

(RP2)  $\pi(h)\mathcal{E}_+ = \mathcal{E}_+$  for every  $h \in H$ .

(RP3) There exists a subspace  $\mathcal{D} \subseteq \mathcal{E}_+ \cap \mathcal{E}^\infty$ , dense in  $\mathcal{E}_+$ , such that  $d\pi(X)\mathcal{D} \subset \mathcal{D}$  for all  $X \in \mathfrak{g}$ .

A typical source of reflection positive representations are the representations  $(\pi_\varphi, \mathcal{H}_\varphi)$  obtained via GNS construction (cf. Theorem A.4) from  $\tau$ -invariant positive definite functions  $\varphi: G \rightarrow B(\mathcal{V})$ , respectively the kernel  $K(x, y) = \varphi(xy^{-1})$ , where  $\mathcal{V}$  is a Hilbert space (cf. Appendix A). If  $G_+ \subseteq G$  is an open subset with  $G_+H = G_+$ , then  $\varphi$  is called *reflection positive for*  $(G, G_+, \tau)$  if the kernel  $Q(x, y) = \varphi(x\tau(y)^{-1})$  is positive definite on  $G_+$ . For  $\mathcal{E} = \mathcal{H}_\varphi$ , the subspace  $\mathcal{E}_+$  is generated by the functions  $K(\cdot, y)v$ ,  $y \in G_+$ ,  $v \in \mathcal{V}$ , and  $\widehat{\mathcal{E}}$  identifies naturally with the Hilbert space  $\mathcal{H}_Q \subseteq \mathcal{V}^{G_+}$  (cf. [NO14a, Prop.1.11], [MNO14, Ex. 5.17a]). If the kernels  $\langle Q(x, y)v, v \rangle$  are smooth for  $v$  in a dense subspace of  $\mathcal{V}$ , then (RP1-3) are readily verified.

If  $\pi$  is a reflection positive representation, then  $\pi_H^c(h) := \widehat{\pi(h)}$  defines a unitary representation of  $H$  on  $\widehat{\mathcal{E}}$ . However, we would like to have a unitary representation  $\pi^c$  of the simply connected Lie group  $G^c$  with Lie algebra  $\mathfrak{g}^c$  on  $\widehat{\mathcal{E}}$  extending  $\pi_H^c$  in such a way that the derived representation is compatible with the operators  $\widehat{id\pi(X)}$ ,  $X \in \mathfrak{g}$ , that we obtain from (RP3) on a dense subspace of  $\widehat{\mathcal{E}}$ . If such a representation exists, then we call  $(\pi, \mathcal{E})$  a *euclidean realization* of the representation  $(\pi^c, \widehat{\mathcal{E}})$  of  $G^c$ . Sufficient conditions for the existence of  $\pi^c$  have been developed in [MNO14]. The prototypical pair  $(G, G^c)$  consists of the euclidean motion group  $\mathbb{R}^d \rtimes O_d(\mathbb{R})$  and the simply connected covering of the Poincaré group  $\mathcal{P}_+^\uparrow = \mathbb{R}^d \rtimes SO_{1,d}(\mathbb{R})_0$ .

In [NO14b] we studied reflection positive one-parameter groups and hermitian contractive semigroups as one key to reflection positivity for more general symmetric Lie groups and their representations. Here a crucial point is that, for every unitary one-parameter group  $U_t^c = e^{itH}$  with  $H \geq 0$  on the Hilbert space  $\mathcal{V}$ , we obtain by  $\varphi(t) := e^{-|t|H}$  a  $B(\mathcal{V})$ -valued function on  $\mathbb{R}$  which is reflection positive for  $(\mathbb{R}, \mathbb{R}_+, -\text{id}_{\mathbb{R}})$  and which leads to a euclidean realization of  $U^c$ . From this we derive that all representations of the  $ax + b$ -group, resp., the Heisenberg group which satisfy the positive spectrum condition for the translation group, resp., the center, possess natural euclidean realizations.

The present note grew out of the attempt to extract the representation theoretic aspects from the discussion of reflection positivity for the circle group  $\mathbb{T}$  in [KL81]. Here we continue our project by exploring reflection positive functions  $\varphi: \mathbb{T} \rightarrow B(\mathcal{V})$  for  $(\mathbb{T}, \mathbb{T}^+, \tau)$ , where  $\tau(z) = z^{-1}$  and  $\mathbb{T}^+$  is a half circle. This leads us naturally to anti-unitary involutions, an aspect that did not show up for triples  $(G, G_+, \tau)$ , where  $G_+$  is a semigroup. We start in Section 2 with a discussion of the integral representation of reflection positive operator-valued functions for  $(\mathbb{T}, \mathbb{T}^+, \tau)$  due to Klein and Landau ([KL81]) and in Section 3 we characterize those unitary one-parameter groups  $(U^c, \mathcal{H})$  which admit euclidean realizations in this context as those for which there exists an anti-unitary involution  $J$  commuting with  $U^c$  (Theorem 3.4). Any such pair  $(J, U^c)$  with  $U_t^c = e^{itH}$  can be encoded in the pair  $(J, \Delta)$ , where  $\Delta = e^{-\beta H}$  is positive selfadjoint with  $J\Delta J = \Delta^{-1}$ , a relation well-known from Tomita-Takesaki theory. Such pairs are closely linked to real standard subspaces, a connection discussed in Section 4. In Section 5 we finally explain how all this connects to KMS states for  $C^*$ -dynamical systems  $(\mathcal{A}, \mathbb{R}, \alpha)$  and conclude with a short discussion of perspectives. To establish the connection with KMS states, we need the concept of a positive definite function on a group  $G$  with values in the space  $\text{Sesq}(\mathcal{V})$  of sesquilinear forms on a real or complex vector space  $\mathcal{V}$ . This concept is briefly developed in Appendix A, where we explain in particular how the GNS construction works in this context. Appendix B provides some tools to obtain integral representations of positive definite functions on convex sets which sharpen some known results in this context.

We hope that this note will prove useful in the further development of the representation theoretic side of reflection positivity under the presence of anti-unitary involutions which occur in many recent constructions in Quantum Field Theory (see in particular [BS04, p. 627], [Bo92], [BGL02]).

## 2. Reflection positivity on the circle group $\mathbb{T}$

Let  $G := \mathbb{T}_\beta := \mathbb{R}/\beta\mathbb{Z}$ , where  $\beta > 0$ , so that  $\mathbb{T}_\beta$  is a circle of length  $\beta$ . We write  $[t] := t + \beta\mathbb{Z}$  for the image of  $t$  in  $\mathbb{T}_\beta$  and write  $\mathbb{T}_\beta^+ := \{[t] \in \mathbb{T}_\beta : 0 < t < \beta/2\}$  for the corresponding semicircle. We further fix the involutive automorphism  $\tau_\beta(z) = z^{-1}$  given by inversion in  $\mathbb{T}_\beta$ . In the following we identify functions on  $\mathbb{T}_\beta$  with  $\beta$ -periodic functions on  $\mathbb{R}$ .

**Definition 2.1.** Let  $\mathcal{V}$  be a Hilbert space. A weak operator continuous function  $\varphi: \mathbb{T}_\beta \rightarrow B(\mathcal{V})$  is called *reflection positive* w.r.t.  $(\mathbb{T}_\beta, \mathbb{T}_\beta^+, \tau_\beta)$  if and only if it is positive definite and the kernel  $(\varphi(t+s))_{0 < t, s < \beta/2}$  is positive definite, which is equivalent to the positive definiteness of the kernel

$$Q_\varphi(t, s) := \varphi\left(\frac{t+s}{2}\right), \quad 0 < t, s < \beta \quad \text{on } ]0, \beta[. \quad (1)$$

**Remark 2.2.** The reflection positivity of  $\varphi$  ensures that the corresponding  $\beta$ -periodic GNS-representation of  $\mathbb{R}$  on the reproducing kernel Hilbert space  $\mathcal{E} := \mathcal{H}_\varphi \subseteq \mathcal{V}^{\mathbb{T}_\beta} \subseteq \mathcal{V}^\mathbb{R}$  by  $\pi_\varphi(t)f := f(\cdot + t)$  is reflection positive with respect to  $(\theta f)(x) = f(-x)$  and the closed subspace  $\mathcal{E}_+$  generated by  $\varphi(-t)v$ ,  $t \in \mathbb{T}_\beta^+$ ,  $v \in \mathcal{V}$  (cf. Proposition A.4). Here  $\tau_\beta(x) = -x$  and  $H = \{0\}$ . Conditions (RP1/2) are obvious, and (RP3) is satisfied because the restriction of  $\varphi$  to  $]0, \beta[$  is automatically smooth (Theorem 2.4). For the basis element  $X = 1 \in \mathbf{L}(\mathbb{T}_\beta) \cong \mathbb{R}$ , the operator  $\widehat{d\pi(X)}$  acts on  $\widehat{\mathcal{E}} \subseteq \mathcal{V}^{\mathbb{T}_\beta^+}$  by  $\frac{d}{dt}$ .

**Example 2.3.** Basic examples of reflection positive functions are given by

$$f_\lambda(t) = e^{-t\lambda} + e^{-(\beta-t)\lambda} = 2e^{-\beta\lambda/2} \cosh((\frac{\beta}{2} - t)\lambda)$$

for  $0 \leq t \leq \beta$  and  $\lambda \geq 0$ . The corresponding kernel  $Q_{f_\lambda}$  is positive definite on all of  $\mathbb{R}$  because  $f_\lambda$  is a Laplace transform of the positive measure  $\delta_\lambda + e^{-\beta\lambda}\delta_{-\lambda}$ . A direct calculation shows that the Fourier series of the  $\beta$ -periodic extension of  $f_\lambda$  to  $\mathbb{R}$  (also denoted  $f_\lambda$ ) is given by

$$f_\lambda(t) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n t / \beta} \quad \text{with} \quad c_n = \frac{2\beta\lambda(1 - e^{-\beta\lambda})}{(\lambda\beta)^2 + (2\pi n)^2}. \quad (2)$$

As  $c_n \geq 0$  for every  $n \in \mathbb{Z}$ , the function  $f_\lambda$  on  $\mathbb{T}_\beta$  is positive definite. This shows that  $f_\lambda$  is reflection positive.

**Theorem 2.4.** ([KL81, Thm. 3.3]) *A  $\beta$ -periodic weak operator continuous function  $\varphi: \mathbb{R} \rightarrow B(\mathcal{V})$  is reflection positive w.r.t.  $(\mathbb{T}_\beta, \mathbb{T}_\beta^+, \tau_\beta)$  if and only if there exists a  $\text{Herm}(\mathcal{V})_+$ -valued measure  $\mu_+$  on  $[0, \infty[$  such that*

$$\varphi(t) = \int_0^\infty e^{-t\lambda} + e^{-(\beta-t)\lambda} d\mu_+(\lambda) \quad \text{for} \quad 0 \leq t \leq \beta. \quad (3)$$

*Then the measure  $\mu_+$  is uniquely determined by  $\varphi$ .*

*Proof.* If  $\varphi$  is given by (3), then the kernel  $\varphi(\frac{x+y}{2})$  is clearly positive definite. To see that  $\varphi$  is a positive definite function of  $\mathbb{T}_\beta$ , we use the Fourier expansion  $f_\lambda(t) = \sum_n c_n(\lambda) e^{2\pi i n t / \beta}$  from (2) to obtain

$$\varphi(t) = \sum_{n \in \mathbb{Z}} e^{2\pi i n t / \beta} \int_0^\infty c_n(\lambda) d\mu_+(\lambda).$$

Now the positivity of the operators  $\int_0^\infty c_n(\lambda) d\mu_+(\lambda)$  shows that  $\varphi$  is positive definite, hence reflection positive.

If, conversely,  $\varphi: [0, \beta] \rightarrow B(\mathcal{V})$  is reflection positive, then it can be written as a Laplace transform  $\varphi = \mathcal{L}(\mu)$  of a  $\text{Herm}(\mathcal{V})_+$ -valued measure  $\mu$  on  $\mathbb{R}$  ([Gl03, Thm. 18.8]; see also

Theorem B.3 below). Since the reflection in  $\beta/2$  is the composition of the reflection  $r(t) = -t$  in 0 and the translation by  $\beta$ , the function  $\varphi$  is also symmetric with respect to  $\frac{\beta}{2}$ . This symmetry requirement is equivalent to  $r_*\mu = e_{-\beta}\mu$  for  $e_\beta(\lambda) = e^{\beta\lambda}$ , so that  $\varphi$  can be written as in (3).  $\square$

**Remark 2.5.** (a) It is often more convenient to work with the measure  $\mu$  on  $\mathbb{R}$  defined by

$$d\mu(\lambda) = d\mu_+(\lambda) + e^{\beta\lambda}d\mu_+(-\lambda), \quad \text{which satisfies} \quad r_*\mu = e_{-\beta}\mu \quad \text{for} \quad r(\lambda) = -\lambda. \quad (4)$$

We thus obtain the description  $\varphi(t) = \int_{\mathbb{R}} e^{-t\lambda} d\mu(\lambda) = \mathcal{L}(\mu)(t)$  of  $\varphi$  as the Laplace transform  $\mathcal{L}(\mu)$ . The existence of these integrals for  $0 \leq t \leq \beta$  only requires that the measure  $\mu$  is finite. We then have  $\mu(\mathbb{R}) = \varphi(0) = \varphi(\beta)$  by (4).

(b) In view of (4), the measure  $\nu := e_{-\beta/2}\mu$  is symmetric. For the Fourier transform  $\widehat{\mu}(z) = \int_{\mathbb{R}} e^{iz\lambda} d\mu(\lambda)$  which is defined on the closed strip  $\overline{\mathcal{D}_\beta} = \{z \in \mathbb{C} : 0 \leq \text{Im } z \leq \beta\}$ , we thus obtain the relations

$$\widehat{\mu}(i\beta - z) = \widehat{\mu}(z), \quad z \in \overline{\mathcal{D}_\beta}, \quad \text{and} \quad \widehat{\nu}(t) = \widehat{\nu}(-t) = \widehat{\mu}\left(t + i\frac{\beta}{2}\right), \quad t \in \mathbb{R}. \quad (5)$$

**Remark 2.6.** (The representation  $U^c$  on  $\widehat{\mathcal{E}}$ ) With the integral representation from Theorem 2.4, we can make the unitary one-parameter group  $U^c$  of the dual group  $(\mathbb{T}_\beta)^c \cong \mathbb{R}$  on  $\widehat{\mathcal{E}}$  more explicit. From Remark 2.2, we recall that  $\widehat{\mathcal{E}}$  can be identified with the reproducing kernel Hilbert space corresponding to the kernel  $\varphi\left(\frac{x+y}{2}\right)$  on the real interval  $[0, \beta]$  (cf. (1)).

(a) In view of Theorem B.3,  $\widehat{\mathcal{E}}$  can be identified with the vector-valued  $L^2$ -space  $L^2(\mathbb{R}, \mu; \mathcal{V})$  with the scalar product  $\langle \xi, \eta \rangle = \int_{\mathbb{R}} \langle d\mu(\lambda) \xi(\lambda), \eta(\lambda) \rangle$ . On this space we have a natural unitary one-parameter group defined by

$$(U_t^c f)(\lambda) := e^{it\lambda} f(\lambda) \quad \text{for} \quad t, \lambda \in \mathbb{R}.$$

Since  $L^2(\mathbb{R}, \mu; \mathcal{V})$  contains the constant functions, we obtain a bounded operator  $j: \mathcal{V} \rightarrow L^2(\mathbb{R}, \mu; \mathcal{V})$ ,  $j(v)(\lambda) := v$ . Then  $j(\mathcal{V})$  is  $U^c$ -cyclic in  $L^2(\mathbb{R}, \mu; \mathcal{V})$  ([Ne98, Lemma III.8]), and we have

$$\langle U_t^c j(v), j(w) \rangle = \int_{\mathbb{R}} e^{it\lambda} \langle d\mu(\lambda) v, w \rangle = \langle \varphi(-it) v, w \rangle \quad \text{for} \quad v, w \in \mathcal{V}, t \in \mathbb{R}, \quad (6)$$

where we have used the analytic continuation of  $\varphi$  to the strip  $\{z \in \mathbb{C} : 0 \leq \text{Re } z \leq \beta\}$  that follows from the integral representation (Theorem 2.4) and Theorem B.3. Therefore  $(U^c, L^2(\mathbb{R}, \mu; \mathcal{V}))$  can be identified with the vector-valued GNS representation corresponding to the positive definite function  $\widehat{\mu}(t) = \varphi(-it)$  on  $\mathbb{R}$  obtained by analytic continuation from  $\varphi$  (cf. Proposition A.4). If  $(Hf)(\lambda) = \lambda f(\lambda)$  is the infinitesimal generator of  $U_t^c = e^{itH}$ , then, for every  $v \in \mathcal{V}$ , we have  $v \in \mathcal{D}(e^{-\beta H/2})$  because  $\int_{\mathbb{R}} e^{-\beta\lambda} \langle d\mu(\lambda) v, v \rangle = \langle \varphi(\beta) v, v \rangle$  is finite (Lemma B.4). For  $0 \leq t \leq \beta$  and  $v, w \in \mathcal{V}$ , we now obtain  $\langle \varphi(t) v, w \rangle = \int_{\mathbb{R}} e^{-t\lambda} \langle d\mu(\lambda) v, w \rangle = \langle e^{-tH} v, w \rangle$ , resulting in the dilation formula

$$\varphi(t) = j^* e^{-tH} j \quad \text{for} \quad 0 \leq t \leq \beta. \quad (7)$$

Further,  $(Jf)(\lambda) := e^{-\beta\lambda/2} f(-\lambda)$ , is a unitary involution on  $L^2(\mathbb{R}, \mu; \mathcal{V})$  with  $JHJ = -H$ , and

$$R := e^{\beta H/2} J = J e^{-\beta H/2}, \quad (Rf)(\lambda) = f(-\lambda)$$

is an involution with domain  $\mathcal{D}(R) = \mathcal{D}(e^{-\beta H/2})$  and  $\mathcal{V} \subseteq \text{Fix}(R) := \{v \in \mathcal{D}(R) : Rv = v\}$ .

(b) Alternatively, we can use  $\widetilde{j}: \mathcal{V} \rightarrow L^2(\mathbb{R}, \mu; \mathcal{V})$ ,  $\widetilde{j}(v)(\lambda) = e^{-\beta\lambda/4} v$  to obtain

$$\langle U_t^c \widetilde{j}(v), \widetilde{j}(w) \rangle = \int_{\mathbb{R}} e^{it\lambda} e^{-\beta\lambda/2} \langle d\mu(\lambda) v, w \rangle = \left\langle \varphi\left(\frac{\beta}{2} - it\right) v, w \right\rangle,$$

which exhibits  $U^c$  as the GNS representation of the symmetric positive definite function  $\psi(t) := \varphi\left(\frac{\beta}{2} - it\right)$ .

### 3. Existence of euclidean realizations

The following proposition provides various characterizations of unitary one-parameter groups with reflection symmetry. As we shall see below, these are precisely the ones with a euclidean realization from  $(\mathbb{T}_\beta, \mathbb{T}_\beta^+, \tau_\beta)$ .

**Proposition 3.1.** *For a unitary one-parameter group  $(U_t)_{t \in \mathbb{R}}$  on  $\mathcal{H}$  with spectral measure  $E: \mathfrak{B}(\mathbb{R}) \rightarrow B(\mathcal{H})$ , the following are equivalent:*

- (i) *There exists an anti-unitary involution  $J$  on  $\mathcal{H}$  with  $JU_tJ = U_t$  for  $t \in \mathbb{R}$ .*
- (ii) *For  $\mathcal{H}_\pm := E(\mathbb{R}_\pm^\times)\mathcal{H}$ , the unitary one-parameter groups  $U_t^+ := U_t|_{\mathcal{H}_+}$  and  $U_t^- := U_t|_{\mathcal{H}_-}$  are unitarily equivalent.*
- (iii) *The unitary one-parameter group  $(U, \mathcal{H})$  is equivalent to a GNS representation  $(\pi_\psi, \mathcal{H}_\psi)$ , where  $\psi: \mathbb{R} \rightarrow B(\mathcal{V})$  is a symmetric positive definite function.*
- (iv) *There exists a unitary involution  $R$  on  $\mathcal{H}$  with  $RU_tR = U_{-t}$  for  $t \in \mathbb{R}$ .*

*Proof.* Every cyclic unitary one-parameter group is isomorphic to  $L^2(\mathbb{R}, \mu)$  with  $(U_t f)(\lambda) = e^{it\lambda} f(\lambda)$ . Then  $Kf = \overline{f}$  is an anti-unitary involution. Decomposing into cyclic subspaces, we thus obtain an anti-unitary involution  $K$  on  $\mathcal{H}$  satisfying  $KU_tK = U_{-t}$  for  $t \in \mathbb{R}$ . As (i)-(iv) hold for the trivial representation on the subspace  $\mathcal{H}_0 = E(\{0\})\mathcal{H}$  of fixed points, we may w.l.o.g. assume that  $\mathcal{H}_0 = \{0\}$ , so that  $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ .

(i)  $\Rightarrow$  (ii): The operator  $W := KJ$  is unitary and satisfies  $WU_tW^{-1} = U_{-t}$  for  $t \in \mathbb{R}$ . Therefore  $W\mathcal{H}_\pm = \mathcal{H}_\mp$  and we obtain a unitary intertwining operator  $(U^+, \mathcal{H}_+) \rightarrow (U^-, \mathcal{H}_-)$ .

(ii)  $\Rightarrow$  (iii): We write  $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$  with  $U_t = W_t \oplus W_{-t}$  and  $(W, \mathcal{K}) \cong (U^+, \mathcal{H}_+)$ . Then the representations on both summands are disjoint. Put  $\mathcal{V} := \{(v, v) : v \in \mathcal{K}\}$  and let  $\mathcal{W} \subseteq \mathcal{H}$  be the closed  $U$ -invariant subspace generated by  $\mathcal{V}$ . Since  $\mathcal{W}$  is invariant under all spectral projections, it contains the projection of  $\mathcal{V}$  onto both factors, so that  $\mathcal{W} = \mathcal{H}$ . The corresponding positive definite function  $\psi(t) := P_{\mathcal{V}}U_tP_{\mathcal{V}}^*$  obtained from the orthogonal projection  $P_{\mathcal{V}}: \mathcal{H} \rightarrow \mathcal{V}$  satisfies

$$\langle \psi(t)(v, v), (w, w) \rangle = \langle (W_tv, W_{-t}v), (w, w) \rangle = \langle (W_t + W_{-t})v, w \rangle \quad \text{for } v, w \in \mathcal{K},$$

so that  $\psi$  is symmetric. Hence (iii) follows from Proposition A.4.

(iii)  $\Rightarrow$  (iv): If  $\psi$  is symmetric, then  $R: \mathcal{H}_\psi \rightarrow \mathcal{H}_\psi$ ,  $(Rf)(t) := f(-t)$  is a unitary involution with the required properties because the kernel  $Q(t, s) = \psi(t - s)$  satisfies  $Q(-t, -s) = Q(t, s)$  (cf. [Ne00, Rem. II.4.5(c)]).

(iv)  $\Rightarrow$  (ii) follows from  $R(\mathcal{H}_\pm) = \mathcal{H}_\mp$ .

(ii)  $\Rightarrow$  (i): As above, we write  $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$  with  $U_t = W_t \oplus W_{-t}$  and  $(W, \mathcal{K}) \cong (U^+, \mathcal{H}_+)$  and let  $K: \mathcal{K} \rightarrow \mathcal{K}$  be an anti-unitary involution with  $KU_tK = U_{-t}$  for  $t \in \mathbb{R}$ . Then  $J(v, w) := (Kw, Kv)$  has the required properties.  $\square$

**Remark 3.2.** Let  $U_t = e^{itH}$  be a continuous unitary one-parameter group on the complex Hilbert space  $\mathcal{H}$  with infinitesimal generator  $H$  and  $J$  an anti-unitary involution with  $JU_tJ = U_{\varepsilon t}$  for  $\varepsilon \in \{\pm 1\}$ . Then  $JHJ = -\varepsilon H$ . If the operator  $H$  is non-negative, then so is  $JHJ$ , which can only happen for  $\varepsilon = -1$ .

**Definition 3.3.** We have seen in Remark 2.2 that every weakly operator continuous reflection positive function  $\varphi: \mathbb{T}_\beta \rightarrow B(\mathcal{V})$  leads to a reflection positive unitary representation  $(U, \mathcal{E})$  of  $\mathbb{T}_\beta$  (Remark 2.2). If  $(U^c, \widehat{\mathcal{E}})$  is the corresponding unitary one-parameter group from Remark 2.6, then we call  $(U, \mathcal{E})$  a *euclidean realization of  $U^c$* . We have already seen in Remark 2.6 that it can be obtained by the GNS construction from the positive definite function  $\mathbb{R} \rightarrow B(\mathcal{V})$ ,  $t \mapsto \varphi(it)$  or the symmetric function  $\varphi(it + \frac{\beta}{2})$ . In this sense it is obtained from  $U$  by an analytic continuation process.

At this point it is a natural question which unitary one-parameter groups  $(U^c, \mathcal{H})$  have a euclidean realization in the sense of Definition 3.3. This can now be stated in terms of the conditions discussed in Proposition 3.1:

**Theorem 3.4.** (Realization Theorem) *A unitary one-parameter group  $(U_t^c)_{t \in \mathbb{R}}$  on a Hilbert space  $\mathcal{H}$  has a euclidean realization in terms of a reflection positive representation of  $(\mathbb{T}_\beta, \mathbb{T}_\beta^+, \theta)$  if and only if there exists an anti-unitary involution  $J$  on  $\mathcal{H}$  commuting with  $U^c$ .*

*Proof.* Since the assertion is trivial for the trivial representation, we may assume that there are no non-zero fixed vectors, i.e.,  $\mathcal{H}^{U^c} = \{0\}$ .

From Proposition 3.1 we know that the existence of an anti-unitary involution commuting with  $U^c$  is equivalent to the realizability of  $U^c$  by the GNS construction from a symmetric positive definite function. If  $U^c$  has a euclidean realization as in Definition 3.3, then  $\psi(t) := \varphi(\frac{\beta}{2} + it)$  is such a function (Remark 2.6(b)).

Suppose, conversely, that there exists an anti-unitary involution  $J$  on  $\mathcal{H}$  commuting with  $U^c$ . As in the proof of Proposition 3.1, we write  $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}$  with  $U_t^c = e^{itA} \oplus e^{-itA}$  for a positive selfadjoint operator  $A$  on  $\mathcal{V}$ . For  $\beta > 0$ , we consider the bounded operator  $j: \mathcal{V} \rightarrow \mathcal{H}, j(v) = (v, e^{-\frac{\beta}{2}A}v)$ . Since the projection  $P(v_1, v_2) = (v_1, 0)$  onto the first component is  $E(\mathbb{R}_+^\times)$  for the spectral measure  $E$  of  $U^c$  (here we use that  $E(\{0\}) = 0$ ), the closed  $U^c$ -invariant subspace generated by  $j(\mathcal{V})$  is adapted to the decomposition  $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}$ . Hence its cyclicity follows from the fact that the range of  $e^{-\frac{\beta}{2}A}$  is dense in  $\mathcal{V}$ . As  $j(\mathcal{V})$  is cyclic,

$$\psi: \mathbb{R} \rightarrow B(\mathcal{V}), \quad \psi(t) := j^* U_t^c j = j^* e^{itH} j \quad \text{for} \quad U_t^c = e^{itH},$$

is a positive definite function for which the GNS representation is equivalent to  $(U^c, \mathcal{H})$  (Proposition A.4). Now  $j^*(w_1, w_2) = w_1 + e^{-\frac{\beta}{2}A}w_2$  yields

$$\psi(t) = e^{itA} + e^{-\frac{\beta}{2}A} e^{-itA} e^{-\frac{\beta}{2}A} = e^{itA} + e^{(-it-\beta)A}.$$

Since  $A \geq 0$ ,  $\varphi(z) := \psi(iz) = e^{-zA} + e^{-(\beta-z)A}$  defines a bounded operator for  $0 \leq \operatorname{Re} z \leq \beta$ . For  $0 \leq t \leq \beta$ , we thus obtain the positive definite function

$$\varphi(t) = e^{-tA} + e^{-(\beta-t)A} = j^* e^{-tH} j, \quad 0 \leq t \leq \beta, \quad \text{satisfying} \quad \varphi(\beta - t) = \varphi(t). \quad (8)$$

Hence  $\varphi$  defines a reflection positive function on  $\mathbb{T}_\beta$  for which  $\varphi(-it) = \psi(t)$ , so that by (6) the corresponding representation of the dual group  $(\mathbb{T}_\beta)^c \cong \mathbb{R}$  is equivalent to  $U^c$ .  $\square$

**Remark 3.5.** (a) In [NO14b, Prop. 6.1], we have shown that a unitary one-parameter group  $(U_t)_{t \in \mathbb{R}}$  with  $U_t = e^{itH}$  has a euclidean realization in terms of the triple  $(\mathbb{R}, \mathbb{R}_+, -\operatorname{id}_{\mathbb{R}})$  if and only if  $H \geq 0$ , and then  $\varphi(t) := e^{-|t|H}$  is a corresponding reflection positive function.

(b) For the function  $\varphi$  from (8), a  $\operatorname{Herm}(\mathcal{V})_+$ -valued measure with  $\mathcal{L}(\mu) = \varphi$  is obtained as in (4) from the spectral measure  $E_+$  of  $A$  by  $d\mu(\lambda) = dE_+(\lambda) + e^{\beta\lambda} dE_+(-\lambda)$ . Here  $e^{-tH} = e^{-tA} \oplus e^{tA}$ , so that  $j(v) = (v, e^{-\beta A/2}v) \in \mathcal{D}(e^{\beta H/2})$ .

(c) Let  $U_t^c = e^{itH}$  be a unitary one-parameter group on  $\mathcal{H}$  and  $J$  be a unitary involution on  $\mathcal{H}$  with  $JHJ = -H$ . Then  $R := J e^{-\beta H/2}$  is an unbounded involution with  $\mathcal{D}(R) = \mathcal{D}(e^{-\beta H/2})$ . We further assume that  $j: \mathcal{V} \rightarrow \mathcal{H}$  is a bounded operator with cyclic range contained in  $\operatorname{Fix}(R) = \{v \in \mathcal{D}(R): Rv = v\}$ . Then

$$\varphi(t) := j^* e^{-tH} j \in B(\mathcal{V}), \quad 0 \leq t \leq \beta \quad (9)$$

defines a function on  $[0, \beta]$  for which the kernel  $\varphi(\frac{t+s}{2})$  is positive definite, and we further have

$$\begin{aligned} \varphi(t) &= j^* e^{-tH} j = j^* R^* e^{-tH} R j = j^* e^{-\beta H/2} J e^{-tH} J e^{-\beta H/2} j \\ &= j^* e^{-\beta H/2} e^{tH} e^{-\beta H/2} j = j^* e^{-(\beta-t)H} j = \varphi(\beta - t). \end{aligned}$$

Therefore  $\varphi$  defines a reflection positive function on  $\mathbb{T}_\beta$ . In view of (9), it is the Laplace transform of the measure  $j^*Ej$ , where  $E$  is the spectral measure of  $H$ .

#### 4. Standard subspaces

To connect reflection positivity on  $\mathbb{T}_\beta$  with the modular theory of von Neumann algebras, we now take a closer look at *standard real subspaces* of a complex Hilbert space  $\mathcal{H}$ .

**Definition 4.1.** A closed real subspace  $V \subseteq \mathcal{H}$  is said to be *standard* if  $V \cap iV = \{0\}$  and  $\overline{V + iV} = \mathcal{H}$ . Then we define the corresponding *Tomita operator* on the dense subspace  $V_\mathbb{C} := V + iV$  of  $\mathcal{H}$  by  $S_V(x + iy) := x - iy$  for  $x, y \in V$  (cf. [BGL02]).

**Lemma 4.2.** *If  $V \subseteq \mathcal{H}$  is a standard real subspace and  $S := S_V$ , then*

- (i)  *$S$  is a closed densely defined antilinear involution.*
- (ii) *If  $\Delta := S^*S$  and  $S = J\Delta^{1/2}$  is the polar decomposition of  $S$ , then  $\Delta$  is a positive selfadjoint,  $J$  is an anti-unitary involution and  $J\Delta J = \Delta^{-1}$ .*
- (iii)  *$S^* = JSJ$  is the complex conjugation with respect to the standard subspace  $J(V)$ .*
- (iv) *The real orthogonal complement of  $V$  w.r.t.  $\text{Re}\langle \cdot, \cdot \rangle$  is  $iJ(V)$ . In particular,  $\mathcal{H} = V \oplus iJ(V)$  as a real Hilbert space.*
- (v)  *$J(V) = V^{\perp, \omega}$  for the symplectic form  $\omega := \text{Im}\langle \cdot, \cdot \rangle$ .*

*Proof.* (i), (ii): For  $x, y \in V$  and  $z := x + iy$ , we have

$$\|(z, S(z))\|^2 = \|x + iy\|^2 + \|x - iy\|^2 = 2(\|x\|^2 + \|y\|^2), \quad (10)$$

so that the graph of  $S$  is closed because  $V \oplus V$  is complete, which in turn follows from the closedness of  $V$ . We conclude that  $S$  is a closed densely defined operator on  $\mathcal{H}$ , so that  $S$  has a polar decomposition as in (ii). Since  $\text{im}(S) = V_\mathbb{C}$  is dense,  $J$  is an isometry. With the same arguments as in [BR02, Prop. 2.5.11], it now follows that  $J$  is an anti-unitary involution satisfying  $J\Delta J = \Delta^{-1}$ .

(iii) This further leads to  $S^* = \Delta^{1/2}J = JSJ$ , which shows that  $S^* = S_{J(V)}$  is the complex conjugation corresponding to the standard subspace  $J(V)$ .

(iv) For  $w \in \mathcal{H}$ , we have the following chain of equivalences:

$$\begin{aligned} \text{Re}\langle iV, w \rangle = \{0\} &\Leftrightarrow (\forall v \in V_\mathbb{C}) \text{Re}\langle v, w \rangle = \text{Re}\langle Sv, w \rangle = \text{Re}\langle w, Sv \rangle \\ &\Leftrightarrow (\forall v \in V_\mathbb{C}) \langle v, w \rangle = \langle w, Sv \rangle \Leftrightarrow w \in \mathcal{D}(S^*) \ \& \ S^*w = w \Leftrightarrow w \in J(V). \end{aligned}$$

(v) follows immediately from (iv) because  $iV^{\perp, \omega}$  is the orthogonal complement w.r.t.  $\text{Re}\langle \cdot, \cdot \rangle$ .  $\square$

**Remark 4.3.** By Lemma 4.2, the scalar product  $\langle \cdot, \cdot \rangle$  is real-valued on  $V \times V$  if and only if  $V \subseteq J(V)$ , which is equivalent to  $J(V) = V$ . This happens only if  $S^* = S$ , and this in turn is equivalent to  $\Delta = \mathbf{1}$ , i.e.,  $J = S$ .

**Lemma 4.4.** *Let  $\Delta$  be an injective positive selfadjoint operator on  $\mathcal{H}$  and  $J$  an anti-unitary involution with  $J\Delta J = \Delta^{-1}$ . Then*

- (i)  *$S := J\Delta^{1/2}: \mathcal{D}(\Delta^{1/2}) \rightarrow \mathcal{H}$  is a closed antilinear involution and  $V := \{v \in \mathcal{D}(S): Sv = v\}$  is a standard real subspace of  $\mathcal{H}$  with  $S_V = S$ .*
- (ii) *The selfadjoint operator  $H := \log \Delta$  satisfies  $JHJ = -H$  and  $U_{(t, \varepsilon)} := e^{itH} J^\varepsilon = \Delta^{it} J^\varepsilon$  defines an anti-unitary representation of the direct product group  $\mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$ .*

*Proof.* That  $S$  is an involution follows from  $S^2 = J\Delta^{1/2}J\Delta^{1/2} = \Delta^{-1/2}\Delta^{1/2} = \text{id}_{\mathcal{D}(S)}$ . Further, the closedness of the selfadjoint operator  $\Delta^{1/2}$  implies that  $S$  is closed. Now (i) follows from (10), and (ii) is clear.  $\square$

**Example 4.5.** (Modular data of von Neumann algebras) Tomita–Takesaki theory ([BR02, §2.5]) starts with a cyclic separating vector  $\Omega$  for a von Neumann algebra  $\mathcal{M} \subseteq B(\mathcal{H})$ . Then  $V := \{\overline{M\Omega} : M^* = M \in \mathcal{M}\}$  is a standard real subspace of  $\mathcal{H}$ , and for  $S = S_V$ ,  $\Delta := S^*S$  is the *modular operator* and  $\alpha_t(M) := \Delta^{-it}M\Delta^{it}$  defines a one-parameter group of automorphisms of  $\mathcal{M}$  (the *modular group*). Further, the *modular involution*  $J$  satisfies  $J\mathcal{M}J = \mathcal{M}'$ .

The preceding two lemmas imply that we have a one-to-one correspondence between the following data.

- (a) standard closed subspaces  $V \subseteq \mathcal{H}$ .
- (b) pairs  $(\Delta, J)$ , where  $0 < \Delta = \Delta^*$  and  $J$  is an anti-unitary involution satisfying  $J\Delta J = \Delta^{-1}$ .
- (c) pairs  $(H, J)$ , where  $H = H^*$  and  $J$  is an anti-unitary involution satisfying  $JHJ = -H$ .
- (d) strongly continuous anti-unitary representations of  $\mathbb{R} \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{R}^\times$ .

**Remark 4.6.** Let  $U_t^c = e^{itH}$  be a unitary one-parameter group on  $\mathcal{H}$  and  $J$  be an anti-unitary involution on  $\mathcal{H}$  with  $JHJ = -H$ . Then  $S := Je^{-\beta H/2}$  is an unbounded antilinear involution with  $\mathcal{D}(S) = \mathcal{D}(e^{-\beta H/2})$ . We further assume that  $\mathcal{V}$  is a real vector space and that  $j : \mathcal{V} \rightarrow \mathcal{H}$  is a linear map with cyclic range contained in  $\text{Fix}(S) = \{v \in \mathcal{D}(S) : Sv = v\}$ . As the operators  $U_t^c$  commute with  $S$ ,  $\text{Fix}(S)$  is a closed  $U^c$ -invariant subspace. Now  $\varphi(z)(v, w) := \langle e^{-zH}j(v), j(w) \rangle$  defines for  $0 \leq \text{Re } z \leq \beta$  an element  $\varphi(z) \in \text{Sesq}(\mathcal{V})$ . It is easy to see that the kernel  $\varphi(\frac{z+\overline{w}}{2})$  is positive definite. For  $v, w \in \mathcal{V}$ , we further have:

$$\begin{aligned} \varphi(z)(v, w) &= \langle e^{-zH}j(v), j(w) \rangle = \langle e^{-zH}Sj(v), Sj(w) \rangle = \langle e^{-zH}Je^{-\beta H/2}j(v), Je^{-\beta H/2}j(w) \rangle \\ &= \langle Je^{\overline{z}H}e^{-\beta H/2}j(v), Je^{-\beta H/2}j(w) \rangle = \langle e^{-\beta H/2}j(w), e^{\overline{z}H}e^{-\beta H/2}j(v) \rangle \\ &= \langle j(w), e^{-(\beta-\overline{z})H}j(v) \rangle = \overline{\varphi(\beta-\overline{z})(v, w)}. \end{aligned} \quad (11)$$

This means that, for  $0 \leq \text{Re } z \leq \beta$ , the relation  $\varphi(\beta-\overline{z}) = \overline{\varphi(z)}$  holds in  $\text{Sesq}(\mathcal{V})$ . From that we derive in particular that  $\text{Re } \varphi$  defines a reflection positive function  $\mathbb{T}_\beta \rightarrow \text{Herm}(\mathcal{V})$ . Note that all functions  $\varphi^{v,v}(t) := \varphi(t)(v, v)$  are real-valued, hence reflection positive. The function  $\varphi$  itself admits a  $\beta$ -periodic extension to  $\mathbb{R}$  if and only if  $\varphi(0)$  is real-valued on  $V \times V$ . Since the scalar product of  $\mathcal{H}$  is in general not real-valued on  $V \times V$ , the form  $\varphi(0)$  need not be symmetric

## 5. KMS states

Let  $(\mathcal{A}, \mathbb{R}, \alpha)$  be a  $C^*$ -dynamical system, i.e.,  $\alpha : \mathbb{R} \rightarrow \text{Aut}(\mathcal{A})$  is a homomorphism defining a continuous  $\mathbb{R}$ -action on the  $C^*$ -algebra  $\mathcal{A}$ . We recall from [BR96, Props. 5.3.3, 5.3.7] that an  $\alpha$ -invariant state  $\omega$  of  $\mathcal{A}$  is a *KMS state at value*  $\beta > 0$  if, for all  $A, B \in \mathcal{A}$ , there exists a bounded holomorphic function  $F_{A,B} : \mathcal{D}_\beta \rightarrow \mathbb{C}$  extending continuously to  $\overline{\mathcal{D}_\beta}$ , such that

$$F_{A,B}(t) = \omega(A\alpha_t(B)) \quad \text{and} \quad F_{A,B}(t+i\beta) = \omega(\alpha_t(B)A) \quad \text{for } t \in \mathbb{R}. \quad (12)$$

The following lemma links our concept of a positive definite function with values in sesquilinear forms to invariant states of  $\mathcal{A}$  (Definition A.3).

**Lemma 5.1.** (a) *Let  $(\mathcal{A}, G, \alpha)$  be a  $C^*$ -dynamical system and  $\omega$  be a  $G$ -invariant state of  $\mathcal{A}$ . Then*

$$\psi : G \rightarrow \text{Sesq}(\mathcal{A}), \quad \psi(g)(A, B) := \omega(B^*\alpha_g(A))$$

*is a positive definite function.*

*Proof.* Let  $(\pi_\omega, U, \mathcal{H}_\omega, \Omega)$  be the cyclic covariant representation with  $U_g\Omega = \Omega$  for  $g \in G$  ([BR02, Cor. 2.3.17]). For the  $\mathbb{R}$ -equivariant linear map  $j : \mathcal{A} \rightarrow \mathcal{H}_\omega, j(A) = \pi(A)\Omega$  with dense range we then have

$$\psi(g)(A, B) = \langle \pi(B^*\alpha_g(A))\Omega, \Omega \rangle = \langle U_g\pi(A)\Omega, \pi(B)\Omega \rangle = \langle U_gj(A), j(B) \rangle,$$

Therefore the assertion follows from Proposition A.4. □



**Proposition 5.2.** *For  $G = \mathbb{R}$ , an invariant state  $\omega$  is a KMS state if and only if  $\psi$  extends to a function  $\overline{\mathcal{D}_\beta} \rightarrow \text{Sesq}(\mathcal{A})$  which is pointwise continuous and pointwise holomorphic on  $\mathcal{D}_\beta$  on  $\mathcal{A} \times \mathcal{A}$  and satisfies*

$$\psi(t + i\beta)(A, B) = \psi(-t)(B^*, A^*) = \overline{\psi(t)(A^*, B^*)} \quad \text{for } t \in \mathbb{R}. \quad (13)$$

*Proof.* The fact that  $\psi$  is positive definite implies in particular that  $\psi(-t) = \psi(t)^*$ :

$$\psi(-t)(A, B) = \omega(B^* \alpha_{-t}(A)) = \omega(\alpha_t(B^*)A) = \overline{\omega(A^* \alpha_t(B))} = \psi(t)^*(A, B).$$

Comparing with (12), we see that  $\psi(t)(A, B) = F_{B^*, A}(t)$ , so that the assertion follows from  $\omega(\alpha_t(A)B^*) = \omega(A\alpha_{-t}(B^*)) = \psi(-t)(B^*, A^*)$  and [BR96, Props. 5.3.3, 5.3.7].  $\square$

**Remark 5.3.** (a) By analytic continuation, the relation (13) implies that

$$\psi(i\beta - z)(A, B) = \psi(z)(B^*, A^*) = \overline{\psi(-\overline{z})(A^*, B^*)} \quad \text{for } z \in \overline{\mathcal{D}_\beta}.$$

For  $\varphi(t) := \psi(it)$ , we obtain in particular

$$\varphi(\beta - t)(A, B) = \varphi(t)(B^*, A^*) = \overline{\varphi(t)(A^*, B^*)} \quad \text{for } 0 \leq t \leq \beta.$$

Using the notation from Appendix A, we define  $\varphi^{A, B}(t) := \varphi(t)(A, B)$ . Then  $\varphi^{A, A}(t) := \varphi(t)(A, A)$  is real for  $A \in \mathcal{A}$  and that  $\varphi^{A, A^*}(\beta - t) = \varphi^{A, A^*}(t)$ . For  $A = A^*$ , it follows in particular that the function  $\varphi^{A, A}$  on  $[0, \beta]$  defines a reflection positive function on  $\mathbb{T}_\beta$ .

(b) In the situation of the proof of Lemma 5.1(b), for  $G = \mathbb{R}$ , Lemma B.4 implies that  $\pi(\mathcal{A})\Omega \subseteq \mathcal{D}(e^{-\beta H/2})$  and that

$$\varphi(t)(A, B) = \langle e^{-tH} \pi(A)\Omega, \pi(B)\Omega \rangle \quad \text{for } 0 \leq t \leq \beta.$$

(c) In the context of Tomita–Takesaki theory (Example 4.5), we put  $U_t := \Delta^{-it}$  and  $H = -\log \Delta$ , so that  $\Delta^{1/2} = e^{-H/2}$ , which corresponds to  $\beta = 1$ . For the state  $\omega(M) := \langle M\Omega, \Omega \rangle$  and  $\psi(t)(A\Omega, B\Omega) = \langle U_t A\Omega, B\Omega \rangle = \varphi(-it)(A\Omega, B\Omega)$ , we obtain from (11) for  $A = A^*, B = B^*$

$$\psi(t + i)(A\Omega, B\Omega) = \varphi(1 - it)(A\Omega, B\Omega) = \overline{\varphi(-it)(A\Omega, B\Omega)} = \overline{\psi(t)(A\Omega, B\Omega)},$$

which is the KMS condition (13) for  $\beta = 1$  ([BR96, Prop. 5.3.10]).

## 6. Perspectives

The main outcome of the present note is that it clarifies the connection between reflection positivity for the triple  $(\mathbb{T}_\beta, \mathbb{T}_\beta^+, \tau_\beta)$  and the modular data  $(J, \Delta)$  arising naturally in Tomita–Takesaki theory, resp., the theory of KMS states. We hope that this will serve as a basis for a deeper understanding of reflection positivity for higher dimensional groups.

**Relativistic KMS states:** One interesting direction is to connect the relativistic KMS states for a  $d$ -dimensional space time introduced by J. Bros and D. Buchholz in [BB94] to reflection positivity for the group  $\mathbb{R}^d$  (see also [GJ06] for  $d = 2$ ). Here the strip  $\mathcal{D}_\beta \subseteq \mathbb{C}$  is replaced by tube domain

$$\mathcal{T}_\beta := \{z \in \mathbb{C}^d : \text{Im } z \in V_+ \cap (\beta e - V_+)\},$$

where  $V_+$  is the open forward light cone and  $e \in V_+$  is a timelike vector of unit length. One expects a duality between  $G^c = \mathbb{R}^d$  and  $G = \mathbb{T}_\beta \times \mathbb{R}^{d-1}$  and a suitable domain  $G^+ \subseteq G$  for reflection positivity.

**ax+b-group and generalizations:** There are still several aspects of reflection positivity for the  $ax + b$ -group  $G \cong \mathbb{R} \rtimes \mathbb{R}^\times$  (the affine group of the real line) that are not covered by the discussion in [NO14b]. Here the only non-trivial involution is given by  $\tau(b, a) = (-b, a)$ . In [NO14b] we have seen how reflection positivity for  $(G, G^+, \tau)$ , where  $G^+ = \{(b, a) : b > 0, a > 0\}$ , leads to euclidean realizations of those representations  $\pi^c$  of  $G^c \cong \mathbb{R} \rtimes \mathbb{R}_+^\times$  satisfying a positive spectrum condition for translations. For any anti-unitary involution  $J$  satisfying  $J\pi(b, a)J = \pi(-b, a)$ , these representations extend naturally by  $\pi(0, -1) := J$  to anti-unitary representations of the non-connected group  $\mathbb{R} \rtimes \mathbb{R}^\times$ . Such representations arise naturally in the context of Borchers' triples ([BLS11], [Bo92]), where they actually extend to anti-unitary representations of  $G^c := \mathbb{R}^d \rtimes_\gamma \mathbb{R}^\times \cong (\mathbb{R}^2 \times \mathbb{R}^{d-2}) \rtimes \text{SO}_{1,1}(\mathbb{R})$  and  $J = \pi(0, 0, -1)$  acts on Minkowski space  $\mathbb{R}^d \cong \mathbb{R}^2 \oplus \mathbb{R}^{d-2}$  by  $J\pi(b, c, 1)J = \pi(-b, c, 1)$ , where  $\mathbb{R}^2$  is 2-dimensional Minkowski space and  $\mathbb{R}^{d-2}$  is space-like. The dual group is the subgroup  $G \cong (\mathbb{R}^2 \times \mathbb{R}^{d-2}) \rtimes \text{SO}_2(\mathbb{R})$  of the  $d$ -dimensional motion group. This should provide a natural framework for extending the reflection positivity for the circle discussed here in which one can also treat relativistic KMS states. Clearly, the crucial case to be understood is  $d = 2$ . On  $G \cong \mathbb{R}^d \rtimes \text{SO}_2(\mathbb{R})$ , the involution induced by the time reflection  $\theta$  is given by  $\tau(b, a) = (\theta(b), a^{-1})$  and likewise on  $G^c$ . Note that this involution commutes with the action of  $J$ .

**Reflection positivity on the side of  $G^c$ :** Classically, reflection positivity is a condition imposed in the euclidean model where we have a unitary representation  $(\pi, \mathcal{E})$  of the group  $G$ , whereas on the dual side we have a representation  $(\pi^c, \hat{\mathcal{E}})$  which has no specific reflection symmetry. However, in the theory of modular inclusions, one encounters anti-unitary representations of the  $ax + b$ -group corresponding to a modular pair  $(J, \Delta)$  and a unitary one-parameter group  $U$  which satisfy the commutation relations

$$\Delta^{it}U(s)\Delta^{-it} = U(e^{-2\pi t}s) \quad \text{and} \quad JU(s)J = U(-s) \quad \text{for} \quad t, s \in \mathbb{R}.$$

In [BGL02, Thm. 3.2] it is shown that, in this context, the infinitesimal generator  $H$  of  $U(s) = e^{isH}$  satisfies  $H \geq 0$  if and only if  $U(s)V \subseteq V$  holds for  $s \geq 0$ , where  $V$  is the standard real subspace specified by  $(J, \Delta)$  (cf. Lemma 4.4). This situation has a remarkable similarity with the reflection positivity for  $(\mathbb{R}, \mathbb{R}_+, -\text{id}_\mathbb{R})$ , where we have a unitary one-parameter group  $(U_t)_{t \in \mathbb{R}}$  of  $\mathcal{E}$  whose positive part acts by isometries on the subspace  $\mathcal{E}_+$  and the spectrum of the generator of the dual one-parameter group  $U^c$  is positive ([NO14b]). However, here the subspace  $V$  is real,  $\langle Jv, v \rangle \geq 0$  for  $v \in V$  and  $(U(s))_{s \geq 0}$  acts by real isometries on  $V$ .

**More general reflection positive functions:** For a symmetric Lie group  $(G, \tau)$ , the notion of a reflection positive function  $\varphi: G \rightarrow \mathbb{C}$  has been introduced with the idea that these should be positive definite functions corresponding to unitary representations  $U$  on a reflection positive Hilbert space  $(\mathcal{E}, \mathcal{E}_+, \theta)$  such that  $\varphi(g) = \langle \pi(g)v, v \rangle$  holds for a  $\theta$ -fixed vector  $v \in \mathcal{E}_+$ . Then reflection positivity with respect to a domain  $G^+ \subseteq G$  corresponds to  $\pi(G^+)^{-1}v \subseteq \mathcal{E}_+$  (cf. [NO14a]) and is encoded in the positive definiteness of the kernel  $\varphi(g\tau(h)^{-1})$  on  $G^+$ . For  $(\mathbb{R}, \mathbb{R}_+, -\text{id}_\mathbb{R})$  and  $(\mathbb{T}_\beta, \mathbb{T}_\beta^+, \tau_\beta)$ , this implies that  $\varphi$  is real-valued (cf. Theorem 2.4 for  $\dim \mathcal{V} = 1$ ). But this condition is too restrictive to cover the  $\beta$ -periodic functions  $\varphi^{A, A^*}$  from Remarks 4.6 (see also Remark 5.3). This suggests to work with a more general concept where we give up the  $\theta$ -invariance of  $v$ , so that  $\varphi$  need no longer be  $\tau$ -invariant. Extending  $\pi: G \rightarrow \text{U}(\mathcal{E})$  to a representation  $\hat{\pi}$  of the group  $G_\tau := G \rtimes \{\mathbf{1}, \tau\}$  by  $\pi(\mathbf{1}, \tau) := \theta$ , the condition  $\pi(G^+)^{-1}v \subseteq \mathcal{E}_+$  corresponds to the positive definiteness of the kernel  $\langle \theta\pi(h^{-1})v, \pi(g^{-1})v \rangle = \langle \hat{\pi}(g\tau(h)^{-1}, \tau)v, v \rangle$ , which can be expressed in terms of the positive definite function  $\hat{\varphi}(g) := \langle \hat{\pi}(g)v, v \rangle$  on the non-connected group  $G_\tau$ . It is therefore desirable to obtain an explicit description of the so specified functions for  $(\mathbb{R}, \mathbb{R}_+, -\text{id}_\mathbb{R})$  and  $(\mathbb{T}_\beta, \mathbb{T}_\beta^+, \tau_\beta)$  and thus a generalization of the corresponding results in [NO14a] for  $\mathbb{R}$  and Theorem 2.4 for  $\mathbb{T}_\beta$ . Of course, it would be of particular interest to see if the functions  $\varphi^{A, A^*}$  from Remarks 4.6 fall into this larger class.

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## Appendix A. Form-valued kernels

Classically, reproducing kernels arise from Hilbert spaces  $\mathcal{H}$  of functions  $f: X \rightarrow \mathbb{C}$  ( $X$  a set) for which the evaluations  $f \mapsto f(x)$  are continuous, hence representable by elements  $K_x \in \mathcal{H}$  by  $f(x) = \langle f, K_x \rangle$ , and then  $K(x, y) := K_y(x) = \langle K_y, K_x \rangle$  is called the *reproducing kernel* of the space  $\mathcal{H}$ . Then  $K$  determines  $\mathcal{H}$  uniquely. Accordingly, we write  $\mathcal{H}_K \subseteq \mathbb{C}^X$  for the Hilbert space determined by  $K$ . It is a classical result that a kernel function  $K: X \times X \rightarrow \mathbb{C}$  is the reproducing kernel of some Hilbert space if and only if it is *positive definite* in the sense that, for any finite collection  $x_1, \dots, x_n \in X$ , the matrix  $(K(x_j, x_k))_{1 \leq j, k \leq n}$  is positive semidefinite (cf. [Ar50], [Ne00, Ch. 1]). There is a natural generalization to Hilbert spaces  $\mathcal{H}$  of functions with values in a Hilbert space  $\mathcal{V}$ , i.e.,  $\mathcal{H} \subseteq \mathcal{V}^X$ . Then  $K_x(f) = f(x)$  is a linear operator  $K_x: \mathcal{H} \rightarrow \mathcal{V}$  and we obtain a kernel  $K(x, y) := K_x K_y^* \in B(\mathcal{V})$  with values in the bounded operators on  $\mathcal{V}$ . However, there are also situations where one would like to deal with kernels whose values are unbounded operators, so that one has to generalize this context further. As we shall explain below, the concept of a positive definite kernel with values in the space  $\text{Sesq}(\mathcal{V})$  of sesquilinear complex-valued forms on  $\mathcal{V}$  provides a natural context to deal with all relevant cases.<sup>2</sup>

For a real or complex vector space  $\mathcal{V}$ , we write  $\mathcal{V}^\sharp$  for the complex vector space of antilinear maps  $\mathcal{V} \rightarrow \mathbb{C}$ .

**Definition A.1.** Let  $X$  be a set. We call a map  $K: X \times X \rightarrow \text{Sesq}(\mathcal{V})$  a *positive definite kernel* if the associated scalar-valued kernel

$$K^\flat: (X \times \mathcal{V}) \times (X \times \mathcal{V}) \rightarrow \mathbb{C}, \quad K^\flat((x, v), (y, w)) := K(x, y)(w, v)$$

is positive definite. For elements  $f$  of the corresponding reproducing kernel Hilbert space  $\mathcal{H}_{K^\flat} \subseteq \mathbb{C}^{X \times \mathcal{V}}$ , we then have

$$f(y, w) = \langle f, K_{y, w}^\flat \rangle \quad \text{with} \quad K_{y, w}^\flat(x, v) := K^\flat((x, v), (y, w)) = K(x, y)(w, v).$$

Therefore  $v \mapsto K_{x, v}^\flat$  is linear, and this implies that  $f(x, \cdot)$  is antilinear. We identify  $\mathcal{H}_{K^\flat}$  with a subspace of  $(\mathcal{V}^\sharp)^X$  by identifying  $f \in \mathcal{H}_{K^\flat}$  with the function  $f^\sharp: X \rightarrow \mathcal{V}^\sharp$ ,  $f^\sharp(x) := f(x, \cdot)$ . We call

$$\mathcal{H}_K := \{f^\sharp: f \in \mathcal{H}_{K^\flat}\} \subseteq (\mathcal{V}^\sharp)^X$$

the *(vector-valued) reproducing kernel space associated to  $K$* . The elements

$$K_{x, v} := (K_{x, v}^\flat)^\sharp = K(\cdot, x)(v, \cdot), \quad x \in X, v \in \mathcal{V},$$

then form a dense subspace of  $\mathcal{H}_K$  with

$$\langle K_{y, w}, K_{x, v} \rangle = K^\flat((x, v), (y, w)) = K(x, y)(w, v). \quad (\text{A.1})$$

<sup>2</sup> If  $\mathcal{V}$  is a real space, then  $\text{Sesq}(\mathcal{V})$  is the space of bilinear maps  $\mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$ , and if  $\mathcal{V}$  is complex, it stands for those maps  $\mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$  which are linear in the first and antilinear in the second argument. The uniqueness of sesquilinear extension leads to  $\text{Sesq}(\mathcal{V}) = \text{Sesq}(\mathcal{V}_\mathbb{C})$  for a real space  $\mathcal{V}$ . If  $\mathcal{V}$  is complex, then polarization shows that every  $\varphi \in \text{Sesq}(\mathcal{V})$  is uniquely determined by its values on the diagonal.

**Example A.2.** (a) If  $K: X \times X \rightarrow B(\mathcal{V})$  is an operator-valued kernel, where  $\mathcal{V}$  is a complex Hilbert space, then we obtain a  $\text{Sesq}(\mathcal{V})$ -valued kernel by  $Q(x, y)(v, w) := \langle K(x, y)v, w \rangle$ . We have a natural inclusion  $B(\mathcal{V}) \hookrightarrow \text{Sesq}(\mathcal{V})$ ,  $A \mapsto \langle A \cdot, \cdot \rangle$ , whose range is the space of continuous sesquilinear forms. All the functions  $f^\sharp: X \rightarrow \mathcal{V}^\sharp$  in  $\mathcal{H}_Q$  take values in continuous functionals, hence can be identified with  $\mathcal{V}$ -valued functions. This leads to the realization  $\mathcal{H}_K \hookrightarrow \mathcal{V}^X$  (cf. [Ne00, Ch. 1]).

(b) For a one-point set  $X = \{*\}$ , a positive definite  $\text{Sesq}(\mathcal{V})$ -valued kernel is simply a positive semidefinite  $K \in \text{Sesq}(\mathcal{V})$ . The corresponding Hilbert space  $\mathcal{H}_K \subseteq \mathcal{V}^\sharp$  is generated by the elements  $K_v = K(v, \cdot)$  with  $\langle K_v, K_w \rangle = K(v, w)$ .

In particular, if  $\mathcal{V}$  is a Hilbert space, then the natural inclusion  $\mathcal{V} \hookrightarrow \mathcal{V}^\sharp, v \mapsto \langle v, \cdot \rangle$ , corresponds to the kernel  $K(v, w) = \langle v, w \rangle$ .

(c) If  $\mathcal{V} = \mathbb{C}$ , then  $\text{Sesq}(\mathcal{V}) \cong \mathbb{C}$  and  $\text{Sesq}(\mathcal{V})$ -valued kernels are complex-valued kernels.

(d) If  $\mathcal{A}$  is a  $C^*$ -algebra and  $\omega \in \mathcal{A}^*$  a positive functional, then  $K_\omega(A, B) := \omega(B^*A)$  is a positive semidefinite sesquilinear kernel for which the corresponding Hilbert space  $\mathcal{H}_{K_\omega} \subseteq \mathcal{A}^\sharp$  can be obtained from the GNS representation  $(\pi_\omega, \mathcal{H}_\omega, \Omega)$  ([BR02, Cor. 2.3.17]) by the embedding

$$\Gamma: \mathcal{H}_\omega \rightarrow \mathcal{A}^\sharp, \quad \Gamma(\xi)(A) := \langle \xi, \pi(A)\Omega \rangle \quad \text{because} \quad \langle \pi(A)\Omega, \pi(B)\Omega \rangle = \omega(B^*A) = K_\omega(A, B).$$

Note that  $\mathcal{A}$  has a natural representation on  $\mathcal{A}^\sharp$  by  $(A, \beta)(B) := \beta(A^*B)$  and that  $\Gamma$  is equivariant with respect to this representation.<sup>3</sup>

**Definition A.3.** Let  $G$  be a group. A function  $\varphi: G \rightarrow \text{Sesq}(\mathcal{V})$  is said to be *positive definite* if the  $\text{Sesq}(\mathcal{V})$ -valued kernel  $K(g, h) := \varphi(gh^{-1})$  is positive definite.

**Proposition A.4.** (GNS-construction) (a) *Let  $\varphi: G \rightarrow \text{Sesq}(\mathcal{V})$  be a positive definite function. Then  $(\pi_\varphi(g)f)(h) := f(hg)$  defines a unitary representation of  $G$  on the reproducing kernel Hilbert space  $\mathcal{H}_\varphi \subseteq (\mathcal{V}^\sharp)^G$  with kernel  $\varphi(gh^{-1})$  and the range of the map  $j: \mathcal{V} \rightarrow \mathcal{H}_\varphi, j(v)(g) := \varphi(g)(v, \cdot)$  is a cyclic subspace, i.e.,  $\pi(G)j(\mathcal{V})$  spans a dense subspace of  $\mathcal{H}$ . We then have*

$$\varphi(g)(v, w) = \langle \pi_\varphi(g)j(v), j(w) \rangle \quad \text{for } g \in G, v, w \in \mathcal{V}.$$

(b) *If, conversely,  $(\pi, \mathcal{H})$  is a unitary representation of  $G$  and  $j: \mathcal{V} \rightarrow \mathcal{H}$  a linear map whose range is cyclic, then*

$$\varphi: G \rightarrow \text{Sesq}(\mathcal{V}), \quad \varphi(g)(v, w) := \langle \pi(g)j(v), j(w) \rangle$$

*is a  $\text{Sesq}(\mathcal{V})$ -valued positive definite function and  $(\pi, \mathcal{H})$  is unitarily equivalent to  $(\pi_\varphi, \mathcal{H}_\varphi)$ .*

*Proof.* (cf. [Ne00, Sect. 3.1]) (a) For the kernel  $K(g, h) := \varphi(gh^{-1})$  and  $v \in \mathcal{V}$ , the right invariance of the kernel  $K$  implies that  $\pi(g)K_{h,v} = K_{hg^{-1},v}$ , and from that one easily derive the invariance of  $\mathcal{H}_\varphi$  under right translations and the unitarity of their restrictions to  $\mathcal{H}_\varphi$ . Then  $j(v) = \varphi(\cdot)(v, \cdot) = K_{1,v}$  and  $\pi(g)j(v) = K_{g^{-1},v}$  imply that  $j(\mathcal{V}) \subseteq \mathcal{H}_\varphi$  is cyclic. Finally we note that

$$\langle \pi(g)j(v), j(w) \rangle = \langle K_{g^{-1},v}, K_{1,w} \rangle = K(g)(v, w) = \varphi(g)(v, w).$$

(b) The positive definiteness of  $\varphi$  follows easily from the relation  $\varphi(gh^{-1})(v, w) = \langle \pi(h)^{-1}j(v), \pi(g)^{-1}j(w) \rangle$ . Since  $j(\mathcal{V})$  is cyclic, the map

$$\Gamma(\xi)(g)(v) := \langle \xi, \pi(g)^{-1}j(v) \rangle$$

defines an injection  $\mathcal{H} \hookrightarrow (\mathcal{V}^\sharp)^G$  whose range is the subspace  $\mathcal{H}_\varphi$  and which is equivariant with respect to the right translation action  $\pi_\varphi$ .  $\square$

<sup>3</sup> This realization of the Hilbert space  $\mathcal{H}_\omega$  has the advantage that we can see its elements as elements of the space  $\mathcal{A}^\sharp$  (see [Ne00] for many applications of this perspective). Usually,  $\mathcal{H}_\omega$  is obtained as the Hilbert completion of a quotient of  $\mathcal{A}$  by a left ideal which leads to a much less concrete space.

**Remark A.5.** If  $\mathcal{V}$  is a Hilbert space and  $j$  is continuous, then we have the adjoint operator  $j^*: \mathcal{H} \rightarrow \mathcal{V}$  is well-defined and we obtain the  $B(\mathcal{V})$ -valued positive definite function  $\varphi(g) := j^* \pi(g) j$  which can be used to realize  $\mathcal{H}$  in  $\mathcal{V}^G$  (cf. Example A.2(a)).

## Appendix B. Integral representations

For a more concrete realization of unitary representations associated to positive definite functions in  $L^2$ -spaces, integral representations are of crucial importance. The following result is a straightforward generalization of Bochner's Theorem for locally compact abelian groups. Here we write  $\text{Sesq}(\mathcal{V})_+ \subseteq \text{Sesq}(\mathcal{V})$  for the convex cone of positive semidefinite forms.

**Proposition B.1.** *Let  $G$  be a locally compact abelian group. If  $\varphi: G \rightarrow \text{Sesq}(\mathcal{V})$  is a positive definite function for which all functions  $\varphi^{v,w} := \varphi(\cdot)(v, w)$ ,  $v, w \in \mathcal{V}$ , are continuous, then there exists a uniquely determined  $\text{Sesq}(\mathcal{V})_+$ -valued Borel measure  $\mu$  on the locally compact group  $\widehat{G}$  such that  $\widehat{\mu}(g) := \int_{\widehat{G}} \chi(g) d\mu(\chi) = \varphi(g)$  holds for every  $g \in G$  pointwise on  $\mathcal{V} \times \mathcal{V}$ .*

*Proof.* First, Bochner's Theorem for scalar-valued positive definite functions yields for every  $v \in \mathcal{V}$  a finite positive measure  $\mu^v$  on  $\widehat{G}$  such that  $\varphi^{v,v} = \widehat{\mu^v}$ . By polarization, we obtain complex measures  $\mu^{v,w}$ ,  $v, w \in \mathcal{V}$ , on  $\widehat{G}$  with  $\varphi^{v,w} = \widehat{\mu^{v,w}}$ . Then the collection  $(\mu^{v,w})_{v,w \in \mathcal{V}}$  of complex measures on  $\widehat{G}$  defines a  $\text{Sesq}(\mathcal{V})_+$ -valued measure by  $\mu(\cdot)(v, w) := \mu^{v,w}$  for  $v, w \in \mathcal{V}$ , and this measure satisfies  $\widehat{\mu} = \varphi$ .  $\square$

**Remark B.2.** Suppose that  $E$  is the spectral measure on the character group  $\widehat{G}$  for which the continuous unitary representation  $(\pi, \mathcal{H})$  is represented by  $\pi(g) = \int_{\widehat{G}} \chi(g) dE(\chi)$ . Then, for  $v \in \mathcal{H}$ , the positive definite function  $\pi^v(g) := \langle \pi(g)v, v \rangle$  is the Fourier transform of the measure  $E^{v,v} = \langle E(\cdot)v, v \rangle$ . This establishes a close link between spectral measures and the representing measures in the preceding proposition.

We say that a subset  $\mathcal{D}$  in the real vector space  $E$  (contained in  $E_{\mathbb{C}} = E + iE$ ) is *finitely open* if, for each finite-dimensional subspace  $F \subseteq E$ , the intersection  $F \cap \mathcal{D}$  is open in  $F$ . If  $\mathcal{V}$  is a Hilbert space, then a function  $f: T_{\mathcal{D}} := \mathcal{D} + iE \rightarrow E$  on the corresponding tube domain is called *Gateaux holomorphic* if the restriction to each tube domain  $T_{F \cap \mathcal{D}}$  is holomorphic. We write  $\mathcal{O}_G(T_{\mathcal{D}}, \mathcal{V})$  for the space of all Gateaux holomorphic  $\mathcal{V}$ -valued functions on  $T_{\mathcal{D}}$ . For the theory of holomorphic functions on domains in infinite dimensional spaces we refer to Hervé's monograph [He89] or [Ne00] for the connections to representation theory.

**Theorem B.3.** (Laplace transforms and positive definite kernels) *Let  $E$  be a real vector space and  $\mathcal{D} \subseteq E$  a non-empty convex finitely open subset. Let  $\mathcal{V}$  be a Hilbert space and  $\varphi: \mathcal{D} \rightarrow B(\mathcal{V})$  such that*

- (L1) *the kernel  $Q_{\varphi}(x, y) = \varphi(\frac{x+y}{2})$  is positive definite.*
- (L2)  *$\varphi$  is weak operator continuous on every line segment in  $\mathcal{D}$ , i.e., all functions  $t \mapsto \langle \varphi(x + th)v, v \rangle$ ,  $v \in \mathcal{V}$ , are continuous on  $\{t \in \mathbb{R}: x + th \in \mathcal{D}\}$ .*

*Then the following assertions hold:*

- (i) *There exists a unique  $\text{Herm}^+(\mathcal{V})$ -valued measure  $\mu$  on the smallest  $\sigma$ -algebra on  $E^*$  for which all point evaluations are continuous such that*

$$\varphi(x) = \mathcal{L}(\mu)(x) := \int_{E^*} e^{-\lambda(x)} d\mu(\lambda) \quad \text{for } x \in \mathcal{D}.$$

- (ii) *The map*

$$\mathcal{F}: L^2(E^*, \mu; \mathcal{V}) \rightarrow \mathcal{O}_G(T_{\mathcal{D}}, \mathcal{V}), \quad \langle \mathcal{F}(f)(z), v \rangle := \langle f, e_{-\frac{z}{2}} v \rangle$$

is unitary onto the reproducing kernel space  $\mathcal{H}_\varphi := \mathcal{H}_{Q_\varphi}$  corresponding to the kernel associated to  $\varphi$ . It intertwines the unitary representation

$$(\pi(u)f)(\alpha) := e^{i\alpha(u)}f(\alpha) \quad \text{on} \quad L^2(E^*, \mu) \quad \text{and} \quad (\tilde{\pi}(u)f)(z) := f(z - 2iu) \quad \text{on} \quad \mathcal{H}_\varphi.$$

- (iii)  $\varphi$  extends to a unique Gateaux holomorphic function  $\hat{\varphi}$  on the tube domain  $T_{\mathcal{D}}$  which is positive definite in the sense that the kernel  $\hat{\varphi}\left(\frac{z+\overline{w}}{2}\right)$  is positive definite.

*Proof.* (i) follows from [Gl03, Thm. 18.8] and (ii) from [NOr02, Thm. I.7], provided we show that (L2) implies that, for every trace class operator  $S \in B_1(\mathcal{V})$ , the function  $x \mapsto \text{tr}(\varphi(x)S)$  is continuous on line segments in  $\mathcal{D}$ . To this end, we may assume that  $E = \mathbb{R}$  and  $\mathcal{D} = ]a, b[$ . Since  $B_1(\mathcal{H})$  is spanned by positive operators with trace 1, we may w.l.o.g. assume that  $S = S^* \geq 0$  with  $\text{tr} S = 1$ . We may further assume that  $\dim \mathcal{V} = \infty$ ; otherwise the assertion is trivial. Then there exists an orthogonal sequence  $(v_n)_{n \in \mathbb{N}}$  in  $\mathcal{V}$  with  $Sv = \sum_n \langle v, v_n \rangle v_n$ . This leads to

$$\text{tr}(\varphi(x)S) = \sum_n \varphi_n(x) \quad \text{with} \quad \varphi_n(x) := \langle \varphi(x)v_n, v_n \rangle.$$

By assumption, all functions  $\varphi_n$  are continuous. Applying [NOr02, Thm. I.7] to the case  $\mathcal{V} = \mathbb{C}$ , we obtain positive measures  $\nu_n$  on  $\mathbb{R}$  with  $\varphi_n = \mathcal{L}(\nu_n)$  on  $]a, b[$ . Let  $\nu := \sum_n \nu_n$ . Then

$$\mathcal{L}(\nu)(x) = \sum_n \mathcal{L}(\nu_n)(x) = \sum_n \varphi_n(x) = \text{tr}(\varphi(x)S),$$

and the continuity (actually the analyticity) of  $\mathcal{L}(\nu)$  on  $]a, b[$  follows from [Ne00, Cor. V.4.4].  $\square$

The preceding theorem generalizes in an obvious way to  $\text{Sesq}(\mathcal{V})$ -valued functions, where the corresponding measure  $\mu$  has avalues in the cone  $\text{Sesq}(\mathcal{V})_+$ . One can use the same arguments as in the proof of Bochner's Theorem.

The following lemma sharpens the “technical lemma” in [KL83, App. A].

**Lemma B.4.** *Let  $U_t = e^{itH}$  be a unitary one-parameter group on  $\mathcal{H}$ ,  $E$  the corresponding spectral measure,  $v \in \mathcal{H}$ ,  $E^v := \langle E(\cdot)v, v \rangle$ ,  $\beta > 0$  and  $\varphi(t) := \langle U_tv, v \rangle = \int_{\mathbb{R}} e^{it\lambda} dE^v(\lambda)$ . Then the following are equivalent:*

- (i) *There exists a continuous function  $\psi$  on  $\overline{\mathcal{D}_\beta}$ , holomorphic on  $\mathcal{D}_\beta$ , such that  $\psi|_{\mathbb{R}} = \varphi$ .*
- (ii)  $\mathcal{L}(E^v)(\beta) = \int_{\mathbb{R}} e^{-\beta\lambda} dE^v(\lambda) < \infty$ .
- (iii)  $v \in \mathcal{D}(e^{-\frac{\beta}{2}H})$ .

*Proof.* That (i) implies (ii) follows from [Ri66, p. 311]. If, conversely, (ii) is satisfied, then  $\psi(z) := \mathcal{L}(E^v)(-iz)$  is defined on  $\overline{\mathcal{D}_\beta}$ , holomorphic on  $\mathcal{D}_\beta$  and  $\psi|_{\mathbb{R}} = \varphi$ . Finally, the equivalence of (ii) and (iii) follows immediately from the definition of  $e^{-\frac{\beta}{2}H}$  in terms of the spectral measure  $E$ .  $\square$

**Lemma B.5.** (Criterion for the existence of  $\mathcal{L}(\mu)(x)$ ) *Let  $\mathcal{V}$  be a Hilbert space and  $\mu$  be a finite  $\text{Herm}(\mathcal{V})_+$ -valued Borel measure on  $\mathbb{R}$ , so that we can consider its Laplace transform  $\mathcal{L}(\mu)$ , taking values in  $\text{Herm}(\mathcal{V})$ , whenever the integral*

$$\text{tr}(\mathcal{L}(\mu)(x)S) = \int_{\mathbb{R}} e^{-\lambda x} d\mu^S(\lambda) \quad \text{for} \quad d\mu^S(\lambda) = \text{tr}(d\mu(\lambda)S),$$

*exists for every positive trace class operator  $S$  on  $\mathcal{V}$ . This is equivalent to the finiteness of the integrals  $\mathcal{L}(\mu^v)(x)$  for every  $v \in \mathcal{V}$ , where  $d\mu^v(\lambda) = \langle d\mu(\lambda)v, v \rangle$ .*

*Proof.* For  $x \in E$ , the existence of  $\mathcal{L}(\mu)(x)$  implies the finiteness of the integrals  $\mathcal{L}(\mu^v)(x)$  for  $v \in \mathcal{V}$ . Suppose, conversely, that all these integrals are finite. Then we obtain by polarization a hermitian form

$$\beta(v, w) := \int_{\mathbb{R}} e^{-\lambda x} \langle d\mu(\lambda)v, w \rangle$$

on  $\mathcal{V}$ . We claim that  $\beta$  is continuous. As  $\mathcal{V}$  is in particular a Fréchet space, it suffices to show that, for every  $w \in \mathcal{V}$ , the linear functional  $\lambda(v) := \beta(v, w)$  is continuous ([Ru73, Thm. 2.17]).

The linear functionals  $\lambda_n(v) := \int_{-n}^n e^{-\lambda x} \langle d\mu(\lambda)v, w \rangle$  are continuous because  $\mu$  is a bounded measure and the functions  $e_x$  are bounded on bounded intervals. By the Monotone Convergence Theorem, combined with the Polarization Identity,  $\lambda_n \rightarrow \lambda$  holds pointwise on  $\mathcal{V}$ , and this implies the continuity of  $\lambda$  ([Ru73, Thm. 2.8]).

For a positive trace class operators  $S = \sum_n \langle \cdot, v_n \rangle v_n$  with  $\text{tr } S = \sum_n \|v_n\|^2 < \infty$ , we now obtain

$$\mathcal{L}(\mu^S)(x) = \sum_n \mathcal{L}(\mu^{v_n})(x) = \sum_n \beta(v_n, v_n) \leq \|\beta\| \sum_n \|v_n\|^2 < \infty. \quad \square$$

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